Recall how we learned mfd:

- Definition & basic examples
- Constructing mfds
- Associated data (流形的伴随数据) $\begin{cases} \text{vector fields} \\ \text{differential forms} \end{cases}$ ~ sections of vector bundles
- Derivatives (求导)

方向导数的推广

$\begin{cases} \text{derivative w.r.t. connection } \nabla_X(-) \\ \text{exterior derivative } d(-) \leftarrow \text{运算法则} \\ \text{Lie derivative } \mathcal{L}_X(-) \end{cases}$

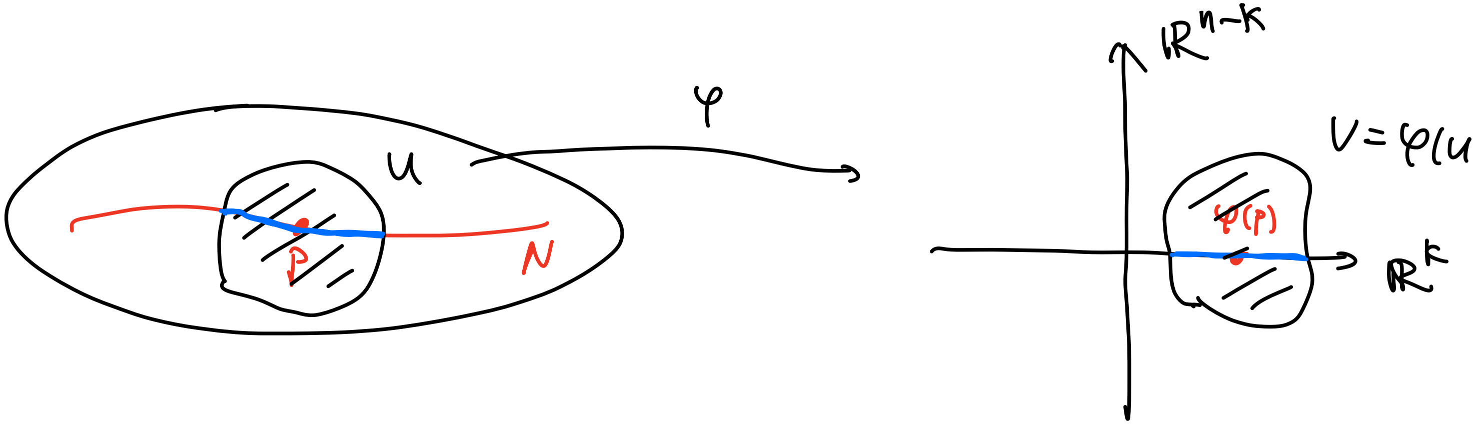
based on pushforward or pullback and flows (相对比较高级)

1. Definition of a submfd

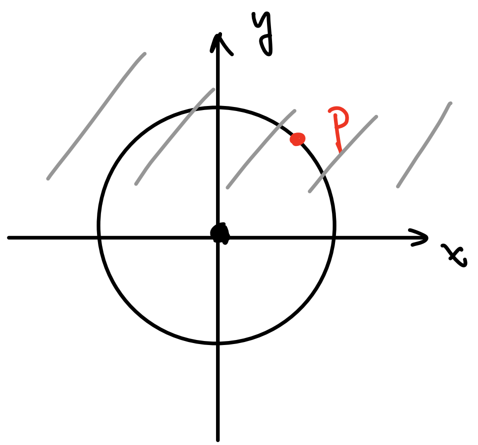
<u>Def</u> Let M^n be a mfd. A subset $N \subset M^n$ is called a (regular or embedded) submfd if $\forall\, p \in N$, $\exists$ a local chart $(U, \varphi : U \xrightarrow{\sim} V)$ of M near p s.t.

$$\varphi(U \cap N) = \{ x \in \underset{\mathbb{R}^n}{\overset{}{V}} \mid x_{k+1} = \cdots = x_n = 0 \}$$

for some $0 \leq k \leq n$. Here, $\dim N = k$ and $n-k$ is called the codimension of N in M.



e.g. $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ is a (reg. or emb.) submfd of $\mathbb{R}^2$, and $\dim S^1 = \operatorname{codim} S^1 \,(\text{in } \mathbb{R}^2) = 1$.



Take $U = \mathbb{R}(x) \times \mathbb{R}(y)_{>0}$

$= $ open upper plane

⊗: $\varphi : U \xrightarrow{\sim} V$ if $\varphi = \mathbb{1}$, then

$\varphi(U \cap S^1) = $ open upper semi-circle

✓: Observe that $U \cap S^1 = $ graph of $f : (-1,1) \to \mathbb{R}^2$

$$= \{ (x, \underbrace{\sqrt{1-x^2}}_{f(x)}) \in \mathbb{R}^2 \mid x \in (-1,1) \}$$

We aim to construct $\varphi : U \xrightarrow{\sim} V$ s.t. $\varphi(\underbrace{(x, \sqrt{1-x^2})}_{\in U \cap S^1}) = (x, \overset{y=0}{0})$

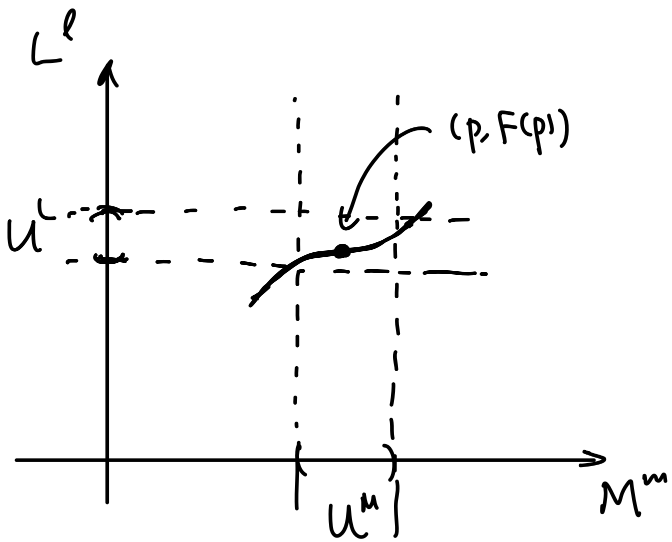
Consider $\varphi : U (\simeq \mathbb{R} \times \mathbb{R}_{>0}) \xrightarrow{\sim} V (\simeq \mathbb{R} \times \mathbb{R}_{>0})$ by

$(x, y) \longrightarrow (x, y - \sqrt{1-x^2})$ ← a homeomorphism e.g. Slightly more general, if $f: \mathbb{R} \to \mathbb{R}$ is a smooth function, then $\text{graph}(f) := \{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ is a dim-1 submfd of $\mathbb{R}^2$.

e.g. If $F: M^m \to L^\ell$ is a smooth map, then

$$\text{graph}(F) = \{(p, F(p)) \in M \times L \mid p \in M\}$$

is a dim-m submfd of $M \times L$.



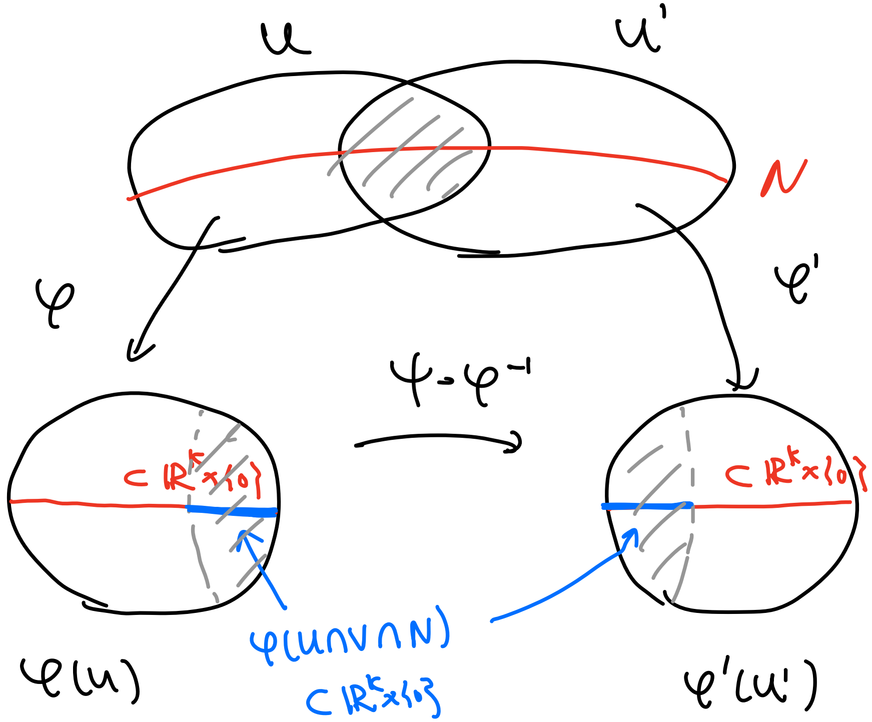
$$U^M \times U^L \xrightarrow[\sim]{\varphi^M \times \varphi^L} V^M \times V^L \subset \mathbb{R}^m \times \mathbb{R}^\ell$$

$$(p, q) \mapsto (\varphi^M(p), \varphi^L(q) - \varphi^L(F(p)))$$

If $q = F(p)$, then 2nd factor above $= 0$.

Note that verifying $N \subset M$ is a submfd is in general not easy.

Prop Any (reg. or emb.) submfd itself is a smooth mfd.



Since $\varphi' \circ \varphi^{-1}$ is smooth, then $\varphi' \circ \varphi^{-1}\big|_{\substack{\text{open subset} \\ \text{in } \mathbb{R}^k \times \{0\}}}$ is also smooth.

e.g. The graph of $f : \mathbb{R} \to \mathbb{R}$ by $x \to |x|$ is <u>not</u> a submfd in $\mathbb{R}^2$.

e.g. $SL(n,\mathbb{R})$ is a submfd of $GL(n,\mathbb{R})$.

Thm (von Neumann, Cartan) Let G be a Lie group. Then any closed subgroup H of G is a (reg. or emb.) Lie subgroup of G.

top algebra submfd and Lie group

$\Rightarrow$ Given Lie groups G and G', and a Lie homomorphism $\varphi: G \to G'$ then $\ker(\varphi)$ is a (reg. or emb.) submfd of G.

e.g. $\Rightarrow$ $\varphi: GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ $\quad A \longrightarrow AA^T$ ($\approx$ polynomials)

This is a Lie homomorphism.

Then $\ker(\varphi) = \{ A \in GL(n,\mathbb{R}) \mid AA^T = \mathbb{1} \} =: O(n)$ is a submfd of $GL(n,\mathbb{R})$.

orthogonal matrix group (正交群)

Question: If a submfd $N (\subset M)$ is already a mfd,

why not study N (independent of M)?

e.g. We have learned a lot on S^n (independent of $S^n \subset \mathbb{R}^{n+1}$).

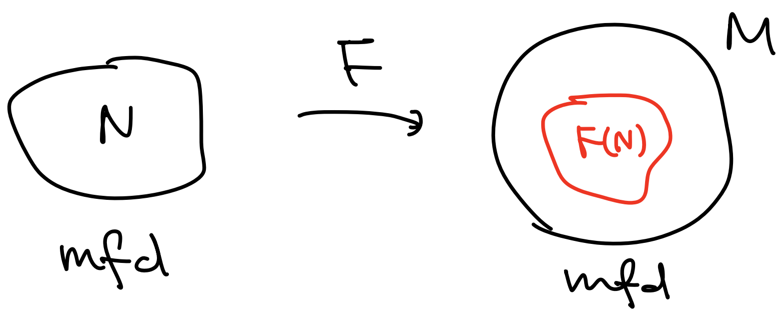
Ans (biased):

ambient mfd

① Submfd(s) often reveal(s) more properties of M.

e.g. Study (cpx) curves in $\mathbb{C}P^n$.

② Often $N \subset M$ is the image of a smooth map

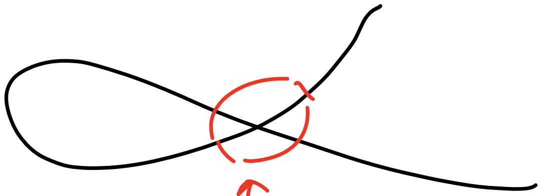


<u>Rmk</u> – The image of a smooth map may be very bad !

e.g. $F: \mathbb{R} \to \mathbb{R}^2$

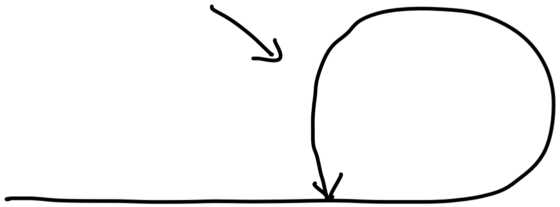
(i)

$\operatorname{im}(F) =$



locally not modelled as a Euclidean space (by looking at it).

speed slows down



$= \operatorname{im}(F)$

$\mathbb{R}$ is <u>not</u> homeomorphic to $\operatorname{im}(F)$, <u>where $\operatorname{im}(F)$ is endowed with subspace top.</u>*

– To give a sufficient condition on $F: N \to M$ s.t. $\operatorname{im}(F)$ is a (reg. or emb.) submfd, one need to use

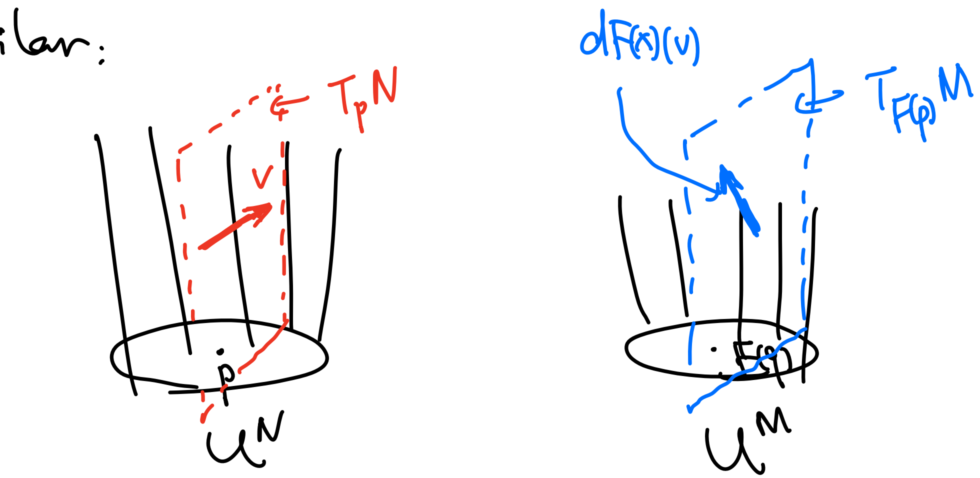
$$F_* = dF : TN \longrightarrow TM.$$

called <u>differential of F</u> or <u>pushforward by F</u>

Recall we have seen pushforward of a diffeo $\varphi : M \circlearrowleft$.

Here, the definition is similar:

$$dF : TN^n \longrightarrow TM^m$$



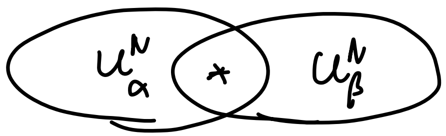
$\left(x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$ $\left(y_1, \cdots, y_m, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_m}\right)$

locally (within local charts)

- on the level of mflds: $p \longrightarrow F(p) = (y_1(p), \cdots, y_m(p))$
- on the level of vectors: $v \longrightarrow \mathrm{Jac}(F) \cdot v = \left(\frac{\partial y_i}{\partial x_j}\right)_{m \times n} \cdot v$

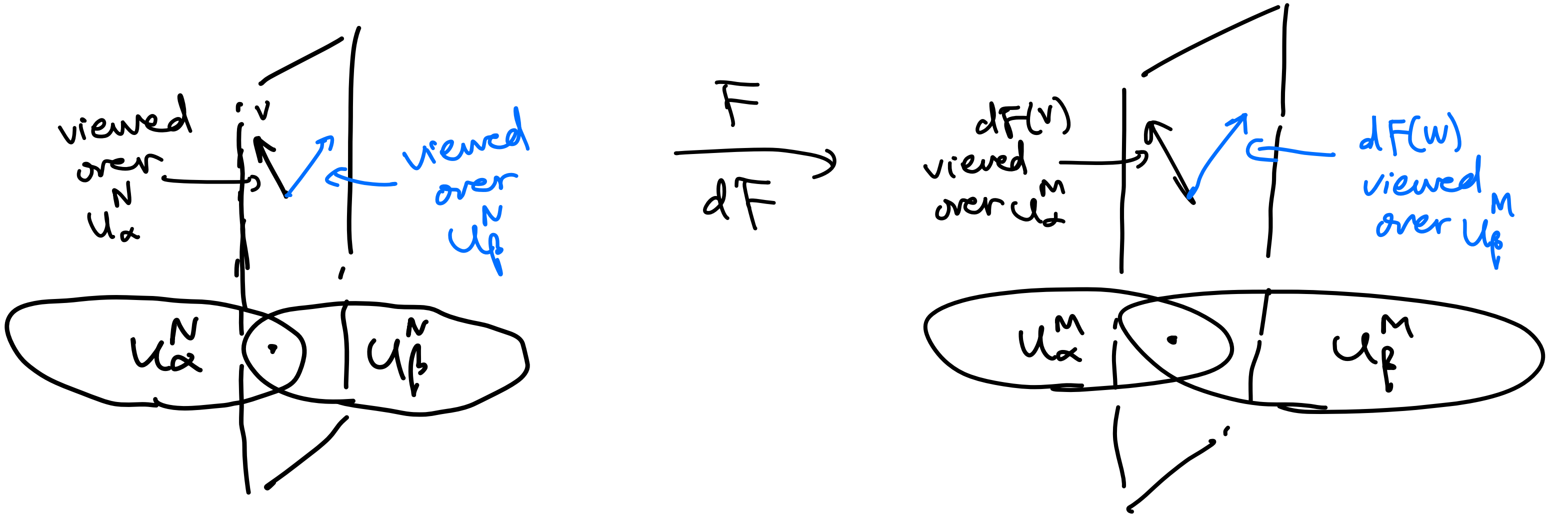
well-defined ?

w.r.t local charts



$\xrightarrow[\varphi_\beta \circ \varphi_\alpha^{-1}]{\text{transition map}}$

$$\begin{cases} \text{mflds:} & p \to p \\ \text{vectors:} & \underbrace{\mathrm{Jac}(\varphi_\beta \circ \varphi_\alpha^{-1}) \cdot (-)}_{TN} \end{cases}$$



$$v \longrightarrow dF(v) = \mathrm{Jac}(F)\, v$$

$$= \mathrm{Jac}^\alpha(F)\, v \longleftarrow (U_\alpha^N, \varphi_\alpha^N) \;\&\; (U_\alpha^M, \varphi_\alpha^M)$$

$$w = \mathrm{Jac}(\varphi_\beta^N \circ \varphi_\alpha^{N^{-1}}) \cdot v \longrightarrow dF(w) = \mathrm{Jac}(F)\, w$$

$$= \mathrm{Jac}^\beta(F)\, w \longleftarrow (U_\beta^N, \varphi_\beta^N) \;\&\; (U_\beta^M, \varphi_\alpha^M)$$

Then $\mathrm{Jac}^\beta(F)\, w = \left(\mathrm{Jac}(\varphi_\beta^M \circ \varphi_\alpha^{M^{-1}}) \cdot \mathrm{Jac}^\alpha(F) \cdot \mathrm{Jac}(\varphi_\alpha^N \circ \varphi_\beta^{N^{-1}})\right) w$

$$= \mathrm{Jac}(\varphi_\beta^M \circ \varphi_\alpha^{M,-1}) \cdot \mathrm{Jac}^\alpha(F)(v) \qquad \checkmark$$

e.g. $F: M^n \longrightarrow \mathbb{R}$ smooth function

$dF: TM^n \longrightarrow T\mathbb{R} \, (= \mathbb{R} \times \overset{\swarrow \text{ vector}}{\mathbb{R}})$ and by def.

$$\underset{\substack{\cap \\ T_pM}}{\overset{\substack{(v_1, \cdots, v_n)^T \\ \shortparallel}}{v}} \longmapsto dF(v) = \mathrm{Jac}(F)v$$

$$= \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}\right)_{1\times n}(v) = \text{directional derivative}$$

<u>Rmk</u> This explains why we denote the basis of tangent space T_pM by $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$.

$$v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}. \implies \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}\right)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\overset{\text{rewrite}}{=} \left(v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}\right) \cdot F = v \cdot F$$

e.g. Let $N^k \subset M^m$ be a (reg. & emb.) submfd. Denote

$$i: N \hookrightarrow M \quad \text{inclusion.}$$

Then over local chart.

$$i : (x_1, \cdots, x_k) \longrightarrow (x_1, \cdots, x_k, 0, \cdots, 0)$$

So $di : TN \to TM \qquad p \to p$ and $v \mapsto \underset{m \times k}{\mathrm{Jac}(i)} \cdot v$

$$= \begin{pmatrix} \boxed{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{k \times k} & 0 \\ 0 & 0 \end{pmatrix} v$$

2. Rank

An important observation:

$\mathrm{Jac}^{\alpha}(F)$ and $\mathrm{Jac}^{\beta}(F)$ in general are not the same.

$$\mathrm{Jac}^{\beta}(F)_{m \times n} = \underbrace{\begin{pmatrix} & \\ & \end{pmatrix}_{m \times m}}_{\text{invertable}} \cdot \mathrm{Jac}^{\alpha}(F)_{m \times n} \cdot \underbrace{\begin{pmatrix} & \\ & \end{pmatrix}_{n \times n}}_{\text{invertible}}$$

BUT

$\Rightarrow$ rank $\mathrm{Jac}^{\alpha}(F) =$ rank $\mathrm{Jac}^{\beta}(F) \quad \forall \alpha, \beta$

$\Rightarrow$ rank of F is a globally defined value (over domain)

e.g. In examples above, rank $(F : M \to \mathbb{R}) = 1$ or 0. rank $(i : N \hookrightarrow M) = k$.